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Asymmetric Vibrations of Polar Orthotropic Laminated Annular Plates

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A theory of general nonsymmetric motion of heterogeneous orthotropic annular plates in terms of radial, circumferential, and transverse displacements is formulated. The eighth-order system of equations obtained generalizes earlier results on axisymmetric vibrations of laminated plates, as well as asymmetric vibrations of symmetrically laminated plates. The eigenvalue problem is solved numerically by the Goodman-Lance method. Accuracy of solution is retained by using the Godunov-Conte criterion of orthonormalizing the base solutions at each point where the least angle between the relevant pair of base vectors is less than a specified tolerance. Numerous examples are presented indicating the effect of plate heterogeneity on the vibration spectrum. An interesting feature is that the density of the eigenfrequencies in laminated plates may be higher than that of the homogeneous plates.

I. Introduction

LITERATURE reviews, including a considerable number of references, on the vibration analysis of homogeneous and heterogeneous orthotropic circular and annular plates are readily available (Leissa,[1] Bert and Francis,[2] and Bert[3]); hence, only a few pertinent papers will be mentioned. Solutions to the free vibration problem of polar orthotropic circular plates were given by Akasaka and Takagishi,[4] Borsuk,[5] and Minkarah and Hoppmann[6] in the form of hypergeometric functions, by the method of Frobenius. For the particular case of isotropic plates, this solution was shown to coincide with that presented by Kirchhoff in terms of Bessel functions. Later, Pandalai and Patel[7] applied a power-series expansion method to the vibration of polar orthotropic circular plates. Axisymmetric vibrations of orthotropic annular plates were analyzed by Pessennikova and Sakharov[8] using the Galerkin method, and by Vijayakumar and Ramaiah[9] using the Rayleigh-Ritz method in conjunction with suitable coordinate transformation. Asymmetric vibrations were considered by Ramaiah and Vijayakumar[10] again by the Rayleigh-Ritz method and by Lizarev, Klenov and Rostanina,[11] using Frobenius' method.

The problem of axisymmetric natural vibrations of *laminated* isotropic circular plates were considered by Stavsky and Loewy.[12] They formulated a sixth-order system of equations of motion in terms of the radial and transverse displacements. It was shown that coupling exists between the extensional and flexural vibrations through the elastic coefficients of the plate and an inertia term. The closed-type solution was obtained in terms of Bessel functions, with an argument determined from a characteristic cubic equation. The problem of axisymmetric natural vibrations of *laminated* orthotropic circular plates,[13] as well as that of asymmetric natural vibrations of symmetrically layered orthotropic annular plates,[14] was solved by Greenberg and Stavsky using the finite-difference method.

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Dedicated to the memory of our late friend and colleague Dr. Nachman Adelman. ז"ל

In the present study, which is an extension of and the logical sequel to Refs. 12-14, the theory of asymmetric motion of general laminated orthotropic annular plates is formulated in terms of radial, circumferential, and transverse displacements. The governing equations are obtained by integrating the elasticity equations. For the special class quasiheterogeneous laminates (including laminates symmetrically oriented about the middle plane), coupling was shown to vanish and the governing equations to reduce to those of a homogeneous orthotropic plate with appropriately chosen stiffness moduli. The theory of axisymmetric vibrations[12] is also included as a particular case of the theory. The eighth-order system of equations obtained is solved numerically by the Godunov-Conte method as described in Refs. 15-17.

II. Formulation of Problem

Consider an annular plate of internal radius r_i and external radius r_o laminated of orthotropic materials bonded together. The elastic properties of each layer are assumed to be generally thickness dependent. Strain-displacement relations are:

$$e_r = \frac{\partial u}{\partial r}, \quad e_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial j}{\partial r}\left(\frac{v}{r}\right) \tag{1}$$

where $u(r,\theta,z)$ and $v(r,\theta,z)$ are the radial and circumferential displacements, respectively, and $w(r,\theta,z)$ is the displacement in the direction normal to the surfaces of bonding ($\equiv$transverse displacement).

Following the basic assumptions of the classical thin plate theory, the transverse displacement is taken to be independent of z, and the radial and circumferential displacements are assumed to be linear functions of the thickness coordinate z:

$$w(r,\theta,z) = w(r,\theta); \quad u(r,\theta,z) = u_o(r,\theta) - z\frac{\partial w(r,\theta)}{\partial r}$$

$$v(r,\theta,z) = v_o(r,\theta) - z\frac{1}{r}\frac{\partial w(r,\theta)}{\partial \theta} \tag{2}$$

Introduction of Eqs. (2) into Eqs. (1) enables us to express the strain components in terms of their values at the reference, $z=0$, and the curvatures

$$(e_r, e_\theta, e_{r\theta}) = (e_{ro}, e_{\theta o}, e_{r\theta o}) + z(\kappa_r, \kappa_\theta, \kappa_{r\theta}) \tag{3}$$

where

$$e_{ro}=\frac{\partial u_o}{\partial r},\ e_{\theta o}=\frac{u_o}{r}+\frac{1}{r}\frac{\partial v_o}{\partial \theta},\ e_{r\theta o}=\frac{1}{r}\frac{\partial u_o}{\partial \theta}+\frac{\partial v_o}{\partial r}-\frac{v_o}{r}$$

$$\kappa_r=-\frac{\partial^2 w}{\partial r^2},\ \kappa_\theta=-\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$\kappa_{r\theta}=-2\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \tag{4}$$

Employing Boussinesq's method,[18] we are converting the infinitesimal dynamic equations to plate equations. (For more detailed derivation of equations see Ref. 19, of which this paper represents an abridged version.) The result in terms of the stress resultants and the stress couples reads:

$$\frac{\partial N_r}{\partial r}+\frac{1}{r}\frac{\partial N_{r\theta}}{\partial\theta}+\frac{N_r-N_\theta}{r}=R_0\ddot{u}_o-R_1\frac{\partial\ddot{w}}{\partial r}$$

$$\frac{\partial N_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial N_\theta}{\partial\theta}+2\frac{N_{r\theta}}{r}=R_0\ddot{v}_o-R_1\frac{1}{r}\frac{\partial\ddot{w}}{\partial\theta}$$

$$\frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta}+\frac{Q_r}{r}=R_0\ddot{w}$$

$$\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_r}{\partial\theta}-Q_r+\frac{M_r-M_\theta}{r}=R_1\ddot{u}_o-R_2\frac{\partial\ddot{w}}{\partial r}$$

$$\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_\theta}{\partial\theta}-Q_\theta+2\frac{M_{r\theta}}{r}=R_1\ddot{v}_o-R_2\frac{1}{r}\frac{\partial w}{\partial\theta} \tag{5}$$

In Eqs. (5), the following definitions were used for the stress resultants, stress couples, and inertia terms:

$$(N_r,N_\theta,N_{r\theta})=\int_{-h_1}^{h_2}(\sigma_r,\sigma_\theta,\tau_{r\theta})\mathrm{d}z$$

$$(Q_r,Q_\theta)=\int_{-h_1}^{h_2}(\tau_{rz},\tau_{\theta z})\mathrm{d}z$$

$$(M_r,M_\theta,M_{r\theta})=\int_{-h_1}^{h_2}(\sigma_r,\sigma_\theta,\tau_{r\theta})z\mathrm{d}z$$

$$(R_0,R_1,R_2)=\int_{-h_1}^{h_2}(1,z,z^2)\rho\mathrm{d}z \tag{6}$$

where h_1 and h_2 are the distances of the bottom and upper faces, respectively, from the reference plate.

Hooke's law for a polar orthothropic layer in plane stress is given by:

$$\sigma_r=E_{rr}e_r+E_{r\theta}e_\theta;\ \sigma_\theta=E_{\theta r}e_r+E_{\theta\theta}e_\theta;\ \tau_{r\theta}=Ge_{r\theta} \tag{7}$$

Relations between the stress resultants and stress couples on the one hand, and the strain components in terms of their values at the reference, $z=0$, and the curvatures become in view of Eqs. (3), (6) and (7):

$$N_r=A_{rr}e_{ro}+A_{r\theta}e_{\theta o}+B_{rr}\kappa_r+B_{r\theta}\kappa_\theta$$

$$N_\theta=A_{r\theta}e_{ro}+A_{\theta\theta}e_{\theta o}+B_{r\theta}\kappa_r+B_{\theta\theta}\kappa_\theta$$

$$N_{r\theta}=A_{ss}e_{r\theta o}+B_{ss}\kappa_{r\theta}$$

$$M_r=B_{rr}e_{ro}+B_{r\theta}e_{\theta o}+D_{rr}\kappa_r+D_{r\theta}\kappa_\theta$$

$$M_\theta=B_{r\theta}e_{ro}+B_{\theta\theta}e_{\theta o}+D_{r\theta}\kappa_r+D_{\theta\theta}\kappa_\theta$$

$$M_{r\theta}=B_{ss}e_{r\theta o}+D_{ss}\kappa_{r\theta} \tag{8}$$

The constants $A_{ij},B_{ij},D_{ij},A_{ss},B_{ss},D_{ss}$ are defined as:

$$(A_{ij},B_{ij},D_{ij})=\int_{-h_1}^{h_2}(1,z,z^2)E_{ij}\mathrm{d}z\quad (i,j=r,\theta)$$

$$(A_{ss},B_{ss},D_{ss})=\int_{-h_1}^{h_2}(1,z,z^2)G\mathrm{d}z \tag{9}$$

Using Eqs. (4) and (8), the plate equations of motion, Eqs. (5), are transformed to the following equations of motion in terms of the reference displacements:

$$L_{11}u_o+L_{12}v_o+L_{13}w=R_0\ddot{u}_o+R_1\ddot{w}_{,r} \tag{10a}$$

$$L_{21}u_o+L_{22}v_o+L_{23}w=R_0\ddot{v}_o-R_1(1/r)\ddot{w}_{,\theta} \tag{10b}$$

$$L_{31}u_o+L_{32}v_o+L_{33}w=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)(R_0u_o-R_2w_{,r})$$
$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left(R_1\ddot{v}_o-\frac{R_2\ddot{w}_{,\theta}}{r}\right)+R_0\ddot{w} \tag{10c}$$

where

$$L_{11}=A_{rr}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)-A_{\theta\theta}\frac{1}{r^2}+A_{ss}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$L_{12}=A_{r\theta}\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}-A_{\theta\theta}\frac{1}{r^2}\frac{\partial}{\partial\theta}+A_{ss}\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\right)$$

$$L_{13}=-B_{rr}\left(\frac{\partial^3}{\partial r^3}+\frac{1}{r}\frac{\partial^2}{\partial r^2}\right)-B_{r\theta}\left(\frac{1}{r^2}\frac{\partial^3}{\partial\theta^2\partial r}-\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2}\right)$$
$$+B_{\theta\theta}\left(\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2}\right)-2B_{ss}\left(\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$

$$L_{21}=A_{ss}\left(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\right)+A_{r\theta}\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}+A_{\theta\theta}\frac{1}{r^2}\frac{\partial}{\partial\theta}$$

$$L_{22}=A_{ss}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)+A_{\theta\theta}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$L_{23}=-2B_{ss}\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta}-B_{r\theta}\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta}$$
$$-B_{\theta\theta}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}+\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3}\right)$$

$$L_{31}=B_{rr}\left(\frac{\partial^3}{\partial r^3}+\frac{2}{r}\frac{\partial^2}{\partial r^2}\right)+B_{r\theta}\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2}$$
$$+B_{\theta\theta}\left(\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r^3}\right)+2B_{ss}\frac{1}{r^2}\frac{\partial^3}{\partial\theta^2\partial r}$$

$$L_{32}=B_{r\theta}\frac{1}{r}\frac{\partial^3}{\partial\theta\partial r^2}+B_{\theta\theta}\left(\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3}-\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}\right.$$
$$\left.+\frac{1}{r^3}\frac{\partial}{\partial\theta}\right)+B_{ss}\frac{2}{r}\frac{\partial^3}{\partial r^2\partial\theta}$$

$$L_{33} = -D_{rr}\left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3}\right) - D_{r\theta}\left(\frac{2}{r^2}\frac{\partial^4}{\partial\theta^2\partial r^2} - \frac{2}{r^3}\frac{\partial^3}{\partial\theta^2\partial r} + \frac{2}{r^4}\frac{\partial^2}{\partial\theta^2}\right) - D_{\theta\theta}\left(\frac{1}{r^4}\frac{\partial^4}{\partial\theta^4} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{2}{r^4}\frac{\partial^2}{\partial\theta^2}\right) - 4D_{ss}\left(\frac{1}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{1}{r^3}\frac{\partial^3}{\partial\theta^2\partial r} + \frac{1}{r^4}\frac{\partial^2}{\partial\theta^2}\right) \tag{11}$$

Different particular cases may be obtained from the eighth-order system Eqs. (10). For the layered plates undergoing axisymmetric vibrations, we have

$$v \equiv 0 \quad \partial u_o/\partial\theta = \partial w/\partial\theta \equiv 0 \tag{12}$$

Equation (10b) turns out to be an identity, and the sixth-order system derived by Stavsky and Loewy[12] for the isotropic layered plates, and that derived by Greenberg and Stavsky[13] for the orthotropic layered plates are obtained.

Consider now a specific class of plates. A composite plate is defined here as quasiheterogeneous if there exists a reference plane with respect to which all moments of the fist order vanish

$$B_{rr}=0,\ B_{r\theta}=B_{\theta r}=0,\ B_{\theta\theta}=0,\ B_{ss}=0,\ R_1=0 \tag{13}$$

Note that an important special case of quasiheterogeneous plates is obtained by symmetrically laminating the annular composite plate; then the reference plane becomes the mid-plane ($h_o = h/2$).

For such specific classes of laminated annular plates, the operators L_{13}, L_{23}, L_{31}, and L_{32} and the inertia terms including R_1 vanish identically and Eqs. (10) become partially uncoupled. Equations (10a) and (10b) reduce to a fourth-order system in terms of in-plane displacements only

$$L_{11}u_o + L_{12}v_o = R_0\ddot{u}_o$$

$$L_{21}u_o + L_{22}v_o = R_0\ddot{v}_o \tag{14}$$

Similarly, Eq. (10c) now becomes a fourth-order equation for the transverse displacement w:

$$L_{33}w = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)R_2 w - \frac{1}{r}\frac{\partial^2}{\partial\theta^2}R_2 w + R_0\ddot{w} \tag{15}$$

Note that the requirement that all statical moments vanish (as in the static case[20]) is not enough for the equations of motion to decouple, and the additional condition $R_1 = 0$ is necessary. Set (14) coincides formally with the equations of extensional vibrations of polar orthotropic disks,[21] whereas Eq. (15) coincides with the equation of flexural vibration of polar-orthotropic plate with rotary inertia included.[22] If the rotary term R_2 is neglected, the classical equation of flexural vibration of polar orthotropic plate is obtained[4-11]:

It is interesting to note that in order to obtain the quasiheterogeneous plate Eqs. (14) and (15), it is enough that elastic moduli and the mass density follow the requirements

$$E_{ij}(z) = E^0_{ij}f(z),\ G(z) = G^0 f(z),\ \rho(z) = \rho^0 f(z) \tag{16a}$$

or

$$E_{ij}(z) = E_{ij}(-z),\ G(z) = G(-z),\ \rho(z) = \rho(-z) \tag{16b}$$

In the general heterogeneous case, we choose the reference plane at a distance

$$h_o = \int_0^h \rho z \mathrm{d}z \Big/ \int_0^h \rho \mathrm{d}z \tag{17}$$

measured from the lower face of the plate, so that R_1 vanishes identically. Note that the lower and upper limits of the integrals in Eqs. (9) are to be taken as $-h_o$ and $h-h_o$, respectively.

The solution of the displacements in Eqs. (10) is expressed in terms of the following normal modes:

$$u_o(r,\theta,t) = U_{o,n}(\xi)\cos n\theta\ \sin\omega t$$

$$v_o(r,\theta,t) = V_{o,n}(\xi)\sin n\theta\ \sin\omega t$$

$$w(r,\theta,t) = W_n(\xi)\cos n\theta\ \sin\omega t,\ \xi \equiv r/r_o \tag{18}$$

Substitution of Eqs. (18) into Eqs. (10) yields the set of ordinary differential equations in terms of $U_{o,n}(\xi)$, $V_{o,n}(\xi)$, and $W_n(\xi)$.[19]

To obtain some insight into the vibrational analysis of composite annular plates, we consider a particular set of boundary conditions, namely those of complete restraint along the edges $r = r_i$ and $r = r_o$, respectively,

$$u_o = v_o = w = 0, \quad \partial w/\partial r = 0 \tag{19}$$

The eigenvalue problem determined by Eqs. (10) and boundary conditions, Eq. (19), is solved numerically by the Godunov-Conte method.

III. Method of Solution

Introducing the following notations,

$$U_{o,n}(\xi) = y_1(\xi) \qquad U'_{o,n}(\xi) = y_2(\xi)$$

$$V_{o,n}(\xi) = y_3(\xi) \qquad V'_{o,n}(\xi) = y_4(\xi)$$

$$W_n(\xi) = y_5(\xi) \qquad W'_n(\xi) = y_6(\xi)$$

$$W''_n(\xi) = y_7(\xi) \qquad W'''_n(\xi) = y_8(\xi) \tag{20}$$

the eigenvalue problem (10) and (19), after eliminating derivatives $\mathrm{d}^2U_{o,n}/\mathrm{d}\xi^2$, $\mathrm{d}^2V_{o,n}/\mathrm{d}\xi^2$, and $\mathrm{d}^3U_{o,n}/\mathrm{d}\xi^3$ in equations which follow after substituting Eqs. (18) into Eqs. (10) and (19) becomes:

$$y' = A(\xi,\Omega)y,\ \eta = r_i/r_o,\ \Omega^2 = \omega^2 r_o^4 R_0/D_{rr} \tag{21a}$$

$$y_1(\eta) = y_3(\eta) = y_5(\eta) = y_6(\eta) = 0 \tag{21b}$$

$$y_1(1) = y_3(1) = y_5(1) = y_6(1) = 0 \tag{21c}$$

where y is the vector function with eight components and A is the 8×8 matrix. Elements of matrix A are given in Ref. 19. The boundary conditions, Eqs. (21b) and (21c) can also be put in matrix form

$$By(\eta) = 0,\ Dy(1) = 0 \tag{22}$$

where B and D are 4×8 matrices:

$$B \equiv D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{23}$$

Following the method of Goodman-Lance,[23] we solve Eq. (21a) four times to obtain solutions $y^{(i)}(\xi,\Omega)$, $i=1,2,3,4$, satisfying the initial conditions

$$[y^{(1)}(\eta,\Omega)]^T = (0,1,0,0,0,0,0,0)$$

$$[y^{(2)}(\eta,\Omega)]^T = (0,0,0,1,0,0,0,0)$$

$$[y^{(3)}(\eta,\Omega)]^T = (0,0,0,0,0,0,1,0)$$

$$[y^{(4)}(\eta,\Omega)]^T = (0,0,0,0,0,0,0,1) \qquad (24)$$

where T denotes the transpose operator.

The conditions, Eqs. (22), are chosen so that

$$BU(\eta,\Omega) = 0 \qquad (25)$$

where $U(\eta,\Omega)$ denotes an eight-by-four matrix whose columns are:

$$y^{(1)}(\eta,\Omega),\ y^{(2)}(\eta,\Omega),\ y^{(3)}(\eta,\Omega),\ \text{and}\ y^{(4)}(\eta,\Omega)$$

The general solution is then given as the linear combination

$$y(\xi,\Omega) = \beta_1 y^{(1)}(\xi,\Omega) + \beta_2 y^{(2)}(\xi,\Omega) + \beta_3 y^{(3)}(\xi,\Omega) + \beta_4 y^{(4)}(\xi,\Omega) \qquad (26)$$

and the boundary conditions Eq. (21b) at $\xi = \eta$ are automatically satisfied. Satisfying the boundary conditions Eq. (21c) at $\xi = 1$ we get

$$\beta_1 y_1^{(1)}(1,\Omega) + \beta_2 y_1^{(2)}(1,\Omega) + \beta_3 y_1^{(3)}(1,\Omega) + \beta_4 y_1^{(4)}(1,\Omega) = 0$$

$$\beta_1 y_3^{(1)}(1,\Omega) + \beta_2 y_3^{(2)}(1,\Omega) + \beta_3 y_3^{(3)}(1,\Omega) + \beta_4 y_3^{(4)}(1,\Omega) = 0$$

$$\beta_1 y_5^{(1)}(1,\Omega) + \beta_2 y_5^{(2)}(1,\Omega) + \beta_3 y_5^{(3)}(1,\Omega) + \beta_4 y_5^{(4)}(1,\Omega) = 0$$

$$\beta_1 y_6^{(1)}(1,\Omega) + \beta_2 y_6^{(2)}(1,\Omega) + \beta_3 y_6^{(3)}(1,\Omega) + \beta_4 y_6^{(4)}(1,\Omega) = 0$$

The conventional requirement of nontriviality yields a characteristic equation:

$$f(\Omega) = \begin{vmatrix} y_1^{(1)}(1,\Omega) & y_1^{(2)}(1,\Omega) & y_1^{(3)}(1/\Omega) & y_1^{(4)}(1,\Omega) \\ y_3^{(1)}(1,\Omega) & y_3^{(2)}(1,\Omega) & y_3^{(3)}(1,\Omega) & y_3^{(4)}(1,\Omega) \\ y_5^{(1)}(1,\Omega) & y_5^{(2)}(1,\Omega) & y_5^{(3)}(1,\Omega) & y_5^{(4)}(1,\Omega) \\ y_6^{(1)}(1,\Omega) & y_6^{(2)}(1,\Omega) & y_6^{(3)}(1,\Omega) & y_6^{(4)}(1,\Omega) \end{vmatrix} = 0 \qquad (28)$$

Equation (28) determines the eigenfrequencies Ω. This procedure, although mathematically exact, may lead to very poor or even completely incorrect results when applied in numerical form. In some cases, the vectors $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, and $y^{(4)}$ become numerically dependent as ξ increases, irrespective of the integration procedure used. The matrix corresponding to Eq. (28) turns out to be poorly conditioned and the eigenvalues Ω are inaccurate. A method avoiding loss of accuracy was proposed by Godunov,[15] whereby the matrix of base solutions remains orthogonal throughout. This is achieved by integrating the set of equations in parallel under the Kronecker-delta initial conditions (which are orthogonal), with the set of solutions reorthogonalized after each integration step by the Gram-Schmidt procedure. The method was subsequently modified and adapted for more practical use by Conte[16] with the angle between the relevant pairs of vectors $y^{(1)}(\xi,\Omega)$, $y^{(2)}(\xi,\Omega)$, $y^{(3)}(\xi,\Omega)$, and $y^{(4)}(\xi,\Omega)$, for the points ξ_i at which the solution is computed—serving as criterion for the need for reorthogonalization. If the least angle is less than a specified tolerance, the solution vectors are to be orthogonalized; otherwise, we proceed to the next step. Conte's test reads:

$$\min_{(i,j)} \cos^{-1} \left| \frac{(y^{(i)}, y^{(j)})}{\{(y^{(i)}, y^{(i)})(y^{(j)}, y^{(j)})\}^{1/2}} \right| < \alpha$$

$$i,j = 1,2,3,4,\quad i \neq j \qquad (29)$$

the parentheses denoting an inner product. For further details see Refs. 15-17.

IV. Numerical Examples and Discussion

To obtain insight into the vibration analysis of laminated annular plates, several examples were investigated. The annual plate, clamped at boundaries, was considered with total thickness $h = 0.04$ in., outer radius-to-thickness ratio $r_o/h = 25$ and inner radius-to-outer radius ratio $r_i/r_o = 0.5$.

In order to check the program, the calculations were performed to compare results with those available in the literature. Results for the homogeneous isotropic plates were practically coincident with those reported by Leissa.[1] Results for symmetrically laminated orthotropic annulus were practically coincident with those of Greenberg and Stavsky[14] obtained by the finite-difference method.

To see the effect on two materials acting in concert, let us consider hybrid plates consisting of an isotropic sheet of thickness h_1 glued to a polar orthotropic sheet of thickness h_2. In Fig. 1 the results are shown for Steel/S-Glass Epoxy hybrids (plates "a") and Aluminum/S-Glass Epoxy layups (plates "b"). The relevant material properties are:

Steel (S), $\rho = 7.492 \times 10^3 \text{kg/m}^3$

$$E = 2.06 \times 10^{11} \text{N/m}^2,\ \nu = 0.3$$

Aluminum (AL), $\rho = 2.395 \times 10^3 \text{kg/m}^3$

$$E = 6.8 \times 10^{10} \text{N/m}^2,\ \nu = 0.3$$

S-Glass Epoxy (SGE), $\rho = 2.003 \times 10^3 \text{kg/m}^3$

$$E_r = 5.17 \times 10^{10} \text{N/m}^2,\ E_\theta = 1.17 \times 10^{10} \text{N/m}^2,$$

$$G = 5.506 \times 10^9 \text{N/m}^2,\ \nu_{r\theta} = 0.25,\ \nu_{\theta r} = 0.0567$$

Elastic stiffness moduli E_{ij} entering in Eqs. (7) are then determined by:

$$E_{rr} = \frac{E_r}{1 - \nu_{r\theta}\nu_{\theta r}},\quad E_{r\theta} = E_{\theta r} = \frac{\nu_{\theta r} E_r}{1 - \nu_{r\theta}\nu_{\theta r}},\quad E_{\theta\theta} = \frac{E_\theta}{1 - \nu_{\theta r}\nu_{r\theta}} \qquad (30)$$

For the isotropic material, we have, correspondingly,

$$E_{rr} = E_{\theta\theta} = \frac{E}{1-\nu^2},\quad E_{r\theta} = E_{\theta r} = \frac{\nu E}{1-\nu^2} \qquad (31)$$

The eigenfrequencies were denoted by $\omega_{n,s}$, where n refers to the number of nodal diameters and s is the number of nodal circles, not including the boundary circles. Note that for homogeneous plates made of either one of the constituents—steel, aluminu, or S-glass epoxy—the natural frequencies $\omega_{0,0}$ are close, but their composition may change their eigenfrequencies. This is shown on Fig. 2, where the eigenfrequency $\omega_{0,0}$ is plotted vs h_1/h and on Fig. 1 where the eigenfrequency $\omega_{0,1}$ is shown again vs h_1/h. As is seen from Fig. 1, the ratio between the extreme values of eigenfrequencies for "a" plates equals 1.27. For "b" plates, the latter is smaller and equals 1.05. The appropriate ratios for the second eigenfrequency of axisymmetrical vibration are 1.43 and 1.41, respectively (Fig. 2).

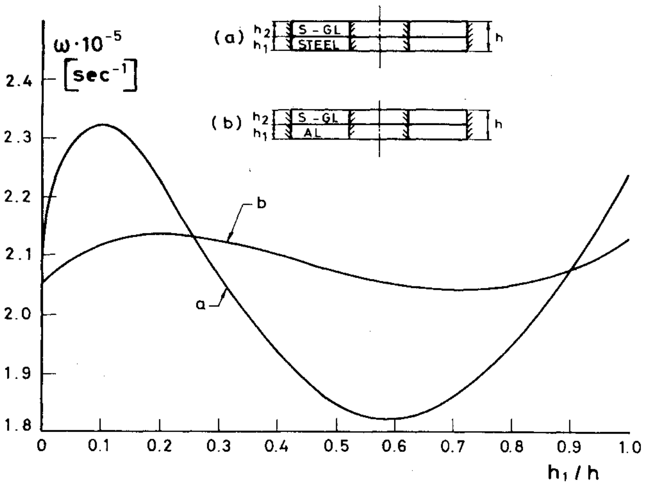

Fig. 1 Variation of eigenfrequency $\omega_{0,0}$ vs h_1/h ratio for two-layer plates.

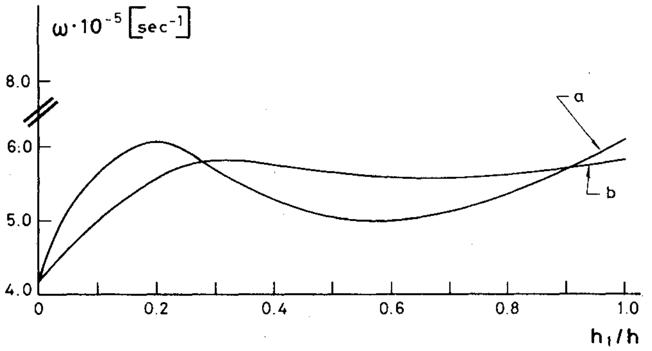

Fig. 2 Variation of eigenfrequency $\omega_{0,1}$ vs h_1/h ratio for two-layer plates.

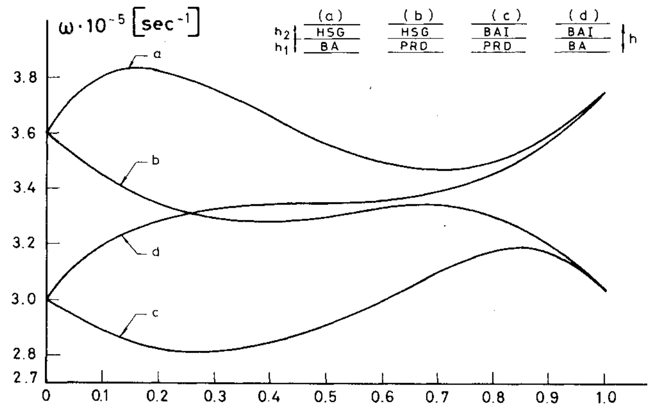

Fig. 3 Influence of material arrangement on $\omega_{0,0}$ for two-layer plates.

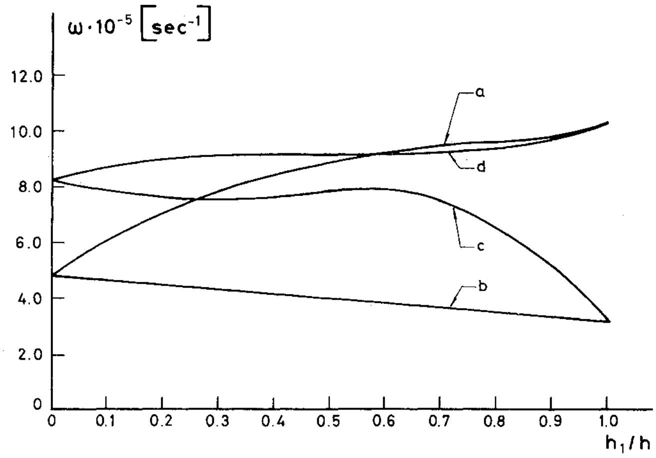

Fig. 4 Influence of material arrangement on $\omega_{0,1}$ for two-layer plates.

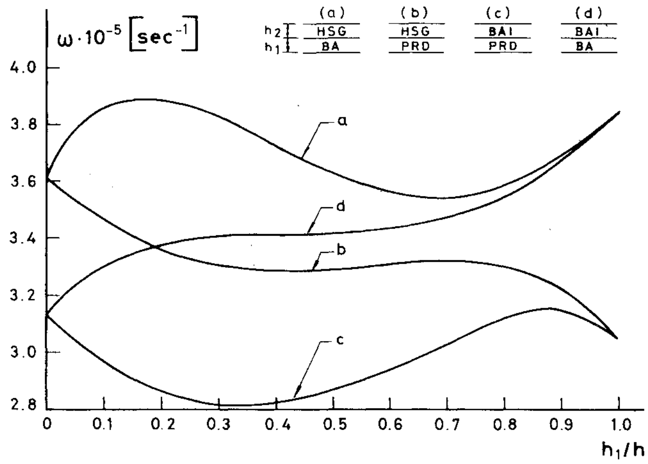

Fig. 5 Dependence of plates' lamination on nonsymmetric vibration frequency $\omega_{2,0}$.

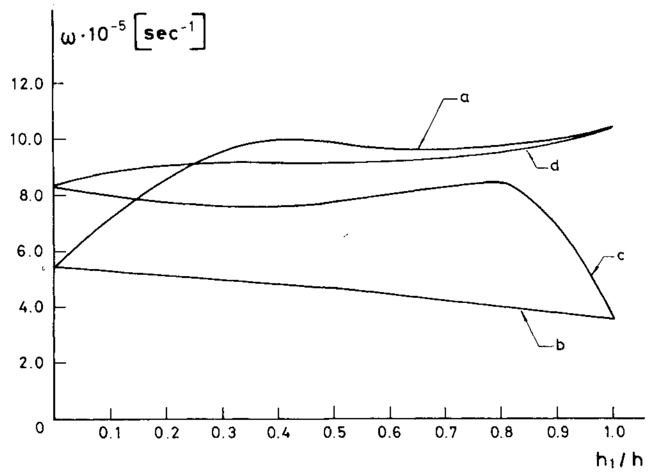

Fig. 6 Dependence of plates' lamination on nonsymmetric vibration frequency $\omega_{2,1}$.

The natural frequencies $\omega_{0,0}$ and $\omega_{0,1}$ are shown on Figs. 3 and 4, respectively for various combinations of two-layered orthotropic annular plates. The laminated plates that were considered are shown in Table 1. The material BAI was obtained formally from BA by interchanging elastic axes. Material properties are as follows:

BA: $\rho = 2.727\times10^3\times\text{kg/m}^3$, $E_r = 2.28\times10^{11}$ N/m^2, $E_\theta = 1.45\times10^{11}$ N/m^2, $G = 4.833\times10^{10}$ N/m^2, $\nu_{r\theta} = 0.23$, $\nu_{\theta r} = 0.1463$

HSGE: $\rho = 1.558\times10^3$ kg/m^3, $E_r = 1.240\times10^{11}$ N/m^2, $E_\theta = 1.03\times10^{10}$ N/m^2, $G = 5.4933\times10^9$ N/m^2, $\nu_{\theta r} = 0.27$, $\nu_{\theta r} = 0.0208$

PRD: $\rho = 1.391\times10^3$ kg/m^3, $E_r = 7.93\times10^{10}$ N/m^2, $E_\theta = 4.1\times10^9$ N/m^2, $G = 2.05\times10^9$ N/m^2, $\nu_{\theta r} = 0.3$, $\nu_{\theta r} = 0.1565$

BAI: $\rho = 2.727\times10^3$ kg/m^3, $E_r = 1.45\times10^{11}$ N/m^2, $E_\theta = 2.28\times10^{11}$ N/m^2, $G = 4.833\times10^{10}$ N/m^2, $\nu_{r\theta} = 0.1463$, $\nu_{\theta r} = 0.23$

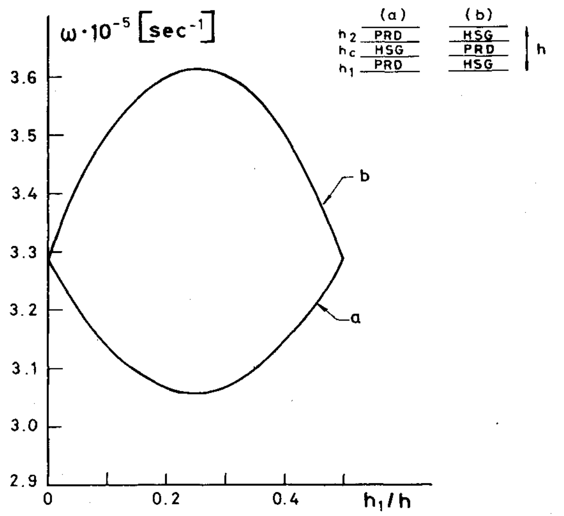

Fig. 7 Variation of $\omega_{0,0}$ for three-layered polar orthotropic laminates vs h_1/h ratio.

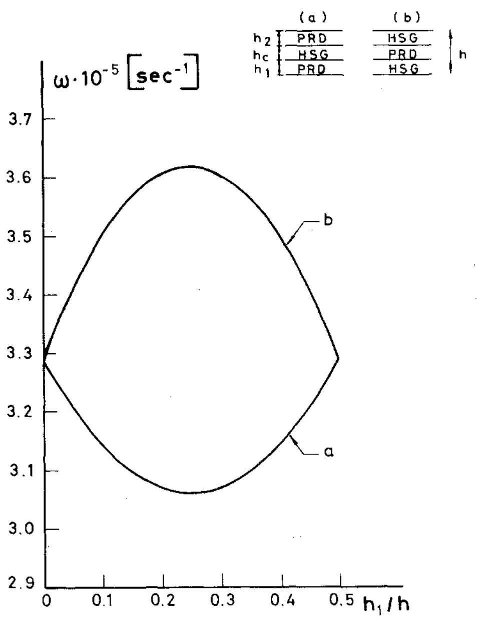

Fig. 8 Variation of $\omega_{1,0}$ for three-layered polar orthotropic laminates vs h_1/h ratio.

As seen in Fig. 3, plates "a" and "c" may also have eigenfrequencies exceeding the maximal eigenfrequency associated with the constituent homogeneous material, whereas for "b" and "d" plates no such phenomenon was observed. The eigenfrequencies with one nodal diameter, also calculated, are not reproduced here because they are very close to those with zero nodal diameter.

Examination of Figs. 3 and 4 suggests that the dependence of the first axisymmetric eigenfrequency on plate layup may be considerably different for the higher natural frequencies. The same phenomenon is observed in Figs. 5 and 6 which portray the behavior of the eigenfrequency $\omega_{2,0}$ and $\omega_{2,1}$, respectively. Figures 7-9 are associated with three-layered plates. These are laminated of outer facings of thicknesses h_1 and h_2, respectively, glued to an inner core of thickness h_c. The outer facings are made of the same material, but different than the core material. In Figs. 7-9, curve "a" shows the frequency variation with h_1/h ratio for PRD facings and HSG core for constant $h_c/h = 0.5$ ratio. Interchanging of the core and facing materials yields in the curve "b" with a maximum value of $\omega_{0,0}$ (Fig. 7), $\omega_{1,0}$ (Fig. 8), and $\omega_{2,0}$ (Fig. 9)

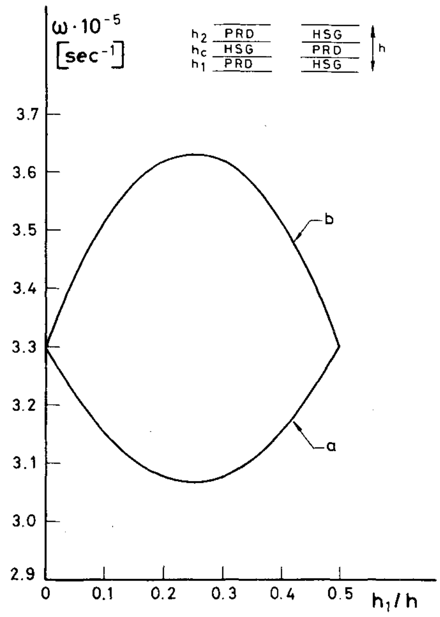

Fig. 9 Variation of $\omega_{2,0}$ for three-layered polar orthotropic laminates vs h_1/h ratio.

Table 1 Characteristics of laminated plates

Laminated plate	Layer 1	Layer 2
a	BA (boron aluminum)	HSGE (high-strength graphite-epoxy)
b	PRD (PRD 49 III)	HSGE
c	PRD	BAI
d	BA	BAI

Table 2 Eigenfrequencies $\omega_{n,0}$ of three-layered plate ($h_1 = 0.3h$, $h_c = 0.5h$, $h_2 = 0.2h$)

n	$\omega_{n,0} \times 10^{-6}$
0	0.309696
1	0.309989
2	0.310903
3	0.312635
4	0.315363
5	0.319411
6	0.325156
7	0.333017
8	0.343402
9	0.356816
10	0.373548
11	0.393927
12	0.418164

at $h_1/h = 0.25$, instead of the minimum, shown in curve "a" for the same h_1/h ratio. In Table 2, the eigenfrequencies $\omega_{n,0}$ are listed for three-layered plate, laminated of outer facings of thicknesses $h_1 = 0.3h$ and $h_3 = 0.2h$ of PRD glued to an inner core made of HSG of thickness $h_c = 0.5h$.

This table illustrates that the density of eigenfrequencies in composite plates considered is higher than that of homogeneous plates. This important property has to be taken into account in the response analysis in such plates and especially in their random vibration study where, due to closeness of the eigenfrequencies at the low end of the frequency domain, the cross correlations between the different modes, become significantly effective.[24-26]

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